AVAILABLE ENERGY IN TOKAMAKS AND STELLARATORS

Per Helander
Max Planck Institute for Plasma Physics
**Motivation**

- Linear stability calculations
  - require that fluctuations be very small
  - may not say much about the eventual fate of the plasma

![Graph showing energy levels](image)

- Would it not also be interesting to calculate $A = E - E_0$?
  - the maximum amount of energy that can be extracted
ENERGY CONSERVATION

- Total energy is conserved

\[ W = E + \Phi = \text{constant} \]

- For the Vlasov-Poisson system

\[ E = \left\langle \sum_a \int \frac{m_a v^2}{2} f_a \, dv \right\rangle, \quad \Phi = \left\langle \frac{\left| \nabla \phi \right|^2}{8\pi} \right\rangle \]

- For collisionless, electrostatic, long-wavelength gyrokinetics

\[ E = \sum_a \left\langle 2\pi B \int f_a(R, \mu, v_\|) \left( \mu B + \frac{m_a v_\|^2}{2} \right) d\mu dv_\| \right\rangle \]

\[ \Phi = \left\langle \frac{\left| \nabla \phi \right|^2}{8\pi} + \sum_a \frac{m_a n_a}{2} \left| \frac{\mathbf{B} \times \nabla \phi}{B^2} \right|^2 \right\rangle \]
Energy Conservation

- Fluctuation energy $\Phi$ is bounded by
  $$\Phi = W - E \leq W - E_0$$

- Ground state nonlinearly stable.
Consider the atmosphere. How much could its potential energy be lowered by adiabatic redistribution of air?

- Available potential energy (Lorenz 1955)
- Usually a few percent
Consider any kinetic equation satisfying Liouville’s theorem

\[ \frac{\partial (\sqrt{g} f)}{\partial t} + \nabla \cdot (\sqrt{g} \dot{x} f) = 0, \]

\[ \frac{\partial (\sqrt{g})}{\partial t} + \nabla \cdot (\sqrt{g} \dot{x}) = 0 \]

What is the lowest accessible energy?

\[ E = \int f \sqrt{g} \, dx \]

Gardner, Phys. Fluids 1963
**GROUND STATE**

- Energy minimised by $F_0[\epsilon(x)]$ with $F_0$ given by the ground-state equation

$$H[F_0(y)] = \Omega(y)$$

a nonlinear integral equation, where

$$\Omega(\kappa) = \int \Theta[\kappa - \epsilon(x)] \sqrt{g} \, dx$$

$$H(\phi) = \int \Theta[f(x) - \phi] \sqrt{g} \, dx$$

- Any monotonically decreasing function of energy alone is a ground state.

- Available energy

$$A = E - E_0 = \int \epsilon(f - F_0) \sqrt{g} \, dx$$

Helander, JPP 2017
EXAMPLE: MAXWELLIAN

- Now choose

\[
f(\mathbf{r}, \mathbf{v}) = n(\mathbf{r}) \left( \frac{m}{2\pi T(\mathbf{r})} \right)^{3/2} e^{\frac{-mv^2}{2T(\mathbf{r})}}
\]

\[
n(\mathbf{r}) = \langle n \rangle [1 + \nu(\mathbf{r})]
\]

\[
T(\mathbf{r}) = \langle T \rangle [1 + \tau(\mathbf{r})]
\]

so that

\[
E = \frac{3V}{2} \langle n \rangle \langle T \rangle \langle 1 + \nu \tau \rangle
\]

- If \( \nu \sim \tau \ll 1 \), then after some algebra we find

\[
\frac{A}{E} = \left\langle \frac{\nu^2}{3} + \frac{\tau^2}{2} \right\rangle
\]
**Density Constraint**

- These ground states have constant density.
  - The distribution function depends on energy alone.

- But sometimes, the density profile is fixed
  - e.g. in turbulence with adiabatic electrons

- Minimisation of energy subject to this constraint gives

\[ f_0(r, v) = F_0[\epsilon(r, v) + \kappa(r)] \]

  - e.g., an isothermal Maxwellian:

\[ f_0 = n(r) \left( \frac{m}{2\pi T} \right)^{3/2} e^{-\frac{mv^2}{2T}}, \quad T = \text{constant} \]

  - General problem: given an initial state, calculate \( F_0 \) and \( \kappa \).
Given an initial state, e.g.,

\[ f(r, v, 0) = M(r) e^{-\frac{mv^2}{2T(r)}} \]

the ground state

\[ f_0(r, v) = F_0[\epsilon(r, v) + \kappa(r)] \]

is determined by two nonlinear integral eqs:

\[
\int F_0[\epsilon + \kappa(r)] dv = n(r)
\]

\[
\int [w(F_0) - \kappa(r)]^{3/2} \Theta [w(F_0) - \kappa(r)] dr = \int \left[ T(r) \ln \frac{M(r)}{F_0} \right]^{3/2} \Theta[M(r) - \phi] dr,
\]

for \( \kappa(r) \) and \( w(F_0) \), the inverse of \( F_0 \).
If the initial state has small density and temperature variations

\[ n(r) = \langle n \rangle [1 + \nu(r)] \]

\[ T(r) = \langle T \rangle [1 + \tau(r)] \]

And the density is required to stay constant the available energy becomes

\[ \frac{A}{E} = \left\langle \frac{\tau^2}{2} \right\rangle \]

In general the available energy from temperature variations however depends on the density profile.
MIXING OF HOT AND COLD PLASMA

- Hot plasma expanding into a cold one:

\[
\begin{array}{|c|c|}
\hline
V_1 & V_2 \\
\hline
n_1, T_1 & n_2, T_2 \\
\hline
\end{array}
\]

- If density evolution is unconstrained and \( T_2 \ll T_1 \)

\[
A = E - E_0 = E \left[ 1 - \left( \frac{V_1}{V_1 + V_2} \right)^{2/3} \right]
\]

- If density cannot change and \( n_2 = 0 \):

\[
A = 0
\]
**Conserved Quantities**

- If there is a conserved quantity, \( \mu \), use phase-space coordinates
  \[
  x = (z, \mu)
  \]

- Then for each \( \mu \), the ground-state equation is as before

\[
H[F_0(\epsilon, \mu), \mu] = \Omega(\epsilon, \mu)
\]

where

\[
H(\phi, \mu) = \int \Theta[f(z, \mu) - \phi] \sqrt{g} \, dz
\]

\[
\Omega(w, \mu) = \int \Theta[w - \epsilon(z, \mu)] \sqrt{g} \, dz
\]

- Any distribution function depending only on \( \epsilon \) and \( \mu \) is a ground state if

\[
\left( \frac{\partial F_0}{\partial \epsilon} \right)_{\mu} \leq 0
\]
The density and temperature of ground states are in general not constant if $B = |B|$ varies

$$\left( \begin{array}{c} n \\ T \end{array} \right) = \pi B \left( \frac{2}{m} \right)^{3/2} \int_0^\infty \left( \frac{1}{2\epsilon} \right) d\epsilon \int_0^{\epsilon/B} \frac{F_0(\epsilon, \mu) d\mu}{\sqrt{\epsilon - \mu B}}$$

Spontaneous peaking of the density and temperature profile may be expected.
If fluctuation frequencies $< \ll$ bounce frequency, the second adiabatic invariant is conserved

$$J = \int m v_{\parallel} \, dl = \text{constant}$$

Ground states:

$$\left( \frac{\partial F_0}{\partial \epsilon} \right)_{\mu, J} \leq 0$$

Now suppose $B$ traces out flux surfaces, $\psi$, and

$$F_0 = f_M(\psi, \epsilon) = n(\psi) \left[ \frac{m}{2\pi T(\psi)} \right]^{3/2} e^{-\epsilon/T(\psi)}$$
GROUND-STATE CRITERION

Now

\[
\left( \frac{\partial f_M}{\partial \epsilon} \right)_{\mu,J} - \frac{f_M}{T} = \frac{f_M}{T} \left( \frac{\omega^T}{\bar{\omega}_d} - 1 \right)
\]

where

\[
\omega_* = \frac{T}{q} \frac{d \ln n}{d \psi}
\]

\[
\omega_* = \omega_* \left[ 1 + \eta \left( \frac{\epsilon}{T} - \frac{3}{2} \right) \right]
\]

\[
\eta = \frac{d \ln T/d \psi}{d \ln n/d \psi}
\]

\[
\omega_d = \mathbf{v}_d \cdot \nabla \alpha
\]

\[
\bar{\omega}_d = \mathbf{v}_d \cdot \nabla \alpha
\]

\[
\mathbf{B} = \nabla \psi \times \nabla \alpha
\]

Ground state if, for all orbits,

\[
\frac{\omega^T}{\bar{\omega}_d} < 1
\]
The precession frequency is geometry-dependent.
- related to average magnetic curvature

 Tokamak
- bad curvature on the outboard side
- good on the inboard side
- cannot satisfy
  \[
  \frac{\omega^T}{\bar{\omega}_d} < 1
  \]
TWO KINDS OF GROUND STATES

There are two ways of satisfying the ground-state criterion

\[ \frac{\omega_*^T}{\bar{\omega}_d} < 1 \]

Either

\[ \omega_* \bar{\omega}_d \leq 0, \quad 0 \leq \eta < \frac{2}{3} \]

or

\[ 0 < \frac{\eta \omega_*}{\bar{\omega}_d} < 1, \quad \eta > \frac{2}{3} \]
FIRST POSSIBILITY: WENDELSTEIN 7-X

Bad curvature

Average good curvature for (almost) all trapped particles

\[ \omega \star \overline{\omega}_d \leq 0 \]
In a device such that

\[ \frac{\partial J}{\partial \psi} < 0 \]

it costs energy to move a particle radially outward:

\[ \Delta J = \frac{\partial J}{\partial \psi} \Delta \psi + \frac{\partial J}{\partial E} \Delta E \quad \Rightarrow \quad \Delta E = -\frac{\partial J/\partial \psi}{\partial J/\partial E} > 0 \]

Linear density-gradient-driven instabilities are then absent.

Rosenbluth (Phys Fluids 1968); Proll, Helander, Connor and Plunk (PRL 2012)
Total heat flux for ITGs with kinetic electrons with varying gradients

\( \frac{Q}{Q_{gb}} \) vs. \( \frac{a}{L_{T_i}} \)

W7-X (red dashed line), DIII-D (blue dotted line)

x5
Effect of density gradient on ITG transport:
NONLINEAR GK SIMULATIONS (PROLL)

- Transport from density-gradient turbulence remarkably low.

![Graph showing total heat flux for TEMs with varying gradients.](image)

Total heat flux for TEMs with varying gradients

- W7-X - grad n
- W7-X - grad T
- HSX - grad n
- HSX - grad T
- DIII-D - grad n
- DIII-D - grad T

with $a/L_{Te}=3$, $a/L_n=0$
**Available Energy of a Flux Tube**

- **Ground-state:**
  
  \[
  \left( \frac{\partial F}{\partial \epsilon} \right)_{\mu, J} = - \frac{f_M}{T} \left| \frac{\omega^T_*}{\omega_d} - 1 \right|
  \]

- **Available energy:**

  \[
  A = \frac{4\pi}{m^2} \int \epsilon(f_M - F)d\psi dJ d\psi d\alpha
  \]

- **Can be evaluated by expanding**

  \[
  f_M(\psi, \mu, J) = f_M(\psi_0, \mu, J) + \frac{\partial f_M}{\partial \psi} (\psi - \psi_0) + \ldots
  \]

  \[
  \epsilon(\psi, \mu, J) = \epsilon(\psi_0, \mu, J) + \frac{\partial \epsilon}{\partial \psi} (\psi - \psi_0) + \ldots
  \]

  \[
  F(\psi, \mu, J) = F(\psi_0, \mu, J) + \frac{\partial f_M}{\partial \psi} (\psi - \psi_0) + \ldots
  \]
AVAILABLE ENERGY OF A FLUX TUBE

- **Result:**

\[
A = \frac{4e^2}{3T}(\Delta\psi)^3 V'(\psi) \int_{\text{tr.}} f_M \bar{\omega}_d^2 h \left( \frac{\omega_*^T}{\bar{\omega}_d} - 1 \right) d^3v
\]

\[
h(x) = x\Theta(x)
\]

- **Scales as**

\[
A_e \sim nT_e \cdot \frac{f_T(\Delta r)^2}{L_n R}
\]

- **If the ions are not subject to J-conservation**

\[
A_i \sim nT_i \left( \frac{\Delta r}{L_n} \right)^2 > A_e
\]

  - Can expect ion-driven instabilities (e.g. Connor, Plunk and Helander, JPP 2017).
2ND CASE: CONFINEMENT BY A LEVITATED COIL
In a point magnetic dipole,

\[ \mathbf{B} = \nabla \psi \times \nabla \alpha \]

\[ \psi = \frac{M}{r} \sin^2 \theta, \quad \alpha = \text{toroidal angle} \]

The precession frequency is, as a function of \( \lambda = \mu/\epsilon \)

\[ \omega_d(\lambda) \geq \omega_d(\lambda = 0) = \frac{2\epsilon}{q\psi} \]

The stability criterion becomes

\[ \frac{2}{3} \frac{d \ln n}{d \ln \psi} < \frac{d \ln T}{d \ln \psi} < 2 \]
FINITE-RADIUS COIL

Two regions separated by the flux surface on which $n$ and $T$ peak:

- **Outer region:**
  - bad curvature
  \[
  \frac{\eta \omega_*}{\bar{\omega}_d} < 1, \quad \eta > \frac{2}{3}
  \]

- **Inner region:**
  - good curvature
  \[
  \omega_*\bar{\omega}_d \leq 0, \quad 0 \leq \eta < \frac{2}{3}
  \]

- Nonlinearly stable plasma.
Stability Diagram

- Linear and nonlinear stability regions

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<table>
<thead>
<tr>
<th>d ln T / d ln Ψ</th>
<th>d ln n / d ln Ψ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable</td>
<td>Unstable</td>
</tr>
</tbody>
</table>
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ANALOGY: ATMOSPHERE

- Stable stratification \((dT/dz > 0)\) of the atmosphere occurs:
  - in the stratosphere,
  - close to the ground during temperature inversion

- No turbulence.
Summary

- Ground state = state of minimum energy under whatever constraints apply
- Available energy = actual energy – ground state energy
- Ground states generally inhomogeneous if adiabatic invariants are conserved.
- Two examples of confined plasmas in ground states:
  - Maximum-J devices (quasi-isodynamic stellarators)
  - Dipole plasmas
- Stable to low-frequency fluctuations of any amplitude.
Proof of nonlinear stability

- The gyrokinetic equations conserve $W = E + \Phi$

\[
E = \sum_a \frac{2\pi}{m_a} \int f_a(R, \mu, v_\parallel) \kappa \sqrt{g} \, dx, \quad \kappa = \mu B + \frac{m_a v_\parallel^2}{2}
\]

\[
\Phi = \int \left( \frac{|\nabla \phi|^2}{8\pi} + \sum_a \frac{m_a n_a}{2} \left| \frac{B \times \nabla \phi}{B^2} \right|^2 \right) \, dx
\]

where for slow fluctuations

\[
E = \sum_a \frac{4\pi}{m_a^2} \int \frac{\kappa f_a \, dl}{|v_\parallel| \tau_b} \, d\mu \, dJ \, d\psi \alpha = \sum_a \frac{4\pi}{m_a^2} \int \epsilon f_a \, d\mu \, dJ \, d\psi \alpha
\]

\[
\epsilon = \frac{1}{\tau_b} \int \frac{\kappa \, dl}{|v_\parallel|}
\]

- Hence

\[
\Phi = W - E \leq W - E_0
\]
**EXAMPLE 1: BI-MAXWELLIAN**

- For a simple Vlasov plasma

\[ \Omega(\epsilon) = \int dr \int_0^\infty \Theta \left( \epsilon - \frac{m v^2}{2} \right) 4\pi v^2 dv = \frac{4\pi V}{3} \left( \frac{2\epsilon}{m} \right)^{3/2} \]

- Bi-Maxwellian initial condition

\[ f(v) = Me^{-\frac{mv^2}{2T_\perp} - \frac{mv^2}{2T_\parallel}} \quad M = n \left( \frac{m}{2\pi \bar{T}} \right)^{3/2} \quad \bar{T} = T_\perp^{2/3} T_\parallel^{1/3} \]

\[ n(r) = \int f dv, \quad T(r) = \frac{2}{3n} \int \epsilon f dv = \frac{2T_\perp + T_\parallel}{3} \]
Level curve $f(v) = \phi$ is an ellipse in velocity space

$$f(v) = Me^{-\frac{m v_{\perp}^2}{2T_\perp} - \frac{m v_{\parallel}^2}{2T_\parallel}} = \phi$$

enclosing the volume

$$H(\phi) = \frac{4\pi V}{3} \left(\frac{2\bar{T}}{m \ln \frac{M}{\phi}}\right)^{3/2}$$

Ground-state equation becomes

$$\frac{4\pi V}{3} \left(\frac{2\bar{T}}{m \ln \frac{M}{F_0}}\right)^{3/2} = \frac{4\pi V}{3} \left(\frac{2\epsilon}{m}\right)^{3/2} \Rightarrow F_0(\epsilon) = Me^{-\epsilon/\bar{T}}$$
The bi-Maxwellian is an equilibrium state but not a ground state.

The ground state is Maxwellian although we have neglected collisions.

The available energy is always positive:

\[ A = \frac{3nV}{2} (T - \bar{T}) \geq 0 \]
SEVERAL PARTICLE SPECIES

- In general, the minimum-energy state will be different for ions and electrons.
  - \( J \) may be conserved only for the electrons if, e.g.,
    \[
    k \parallel v_i \sim \omega \ll k \parallel v_e
    \]

- Quasineutrality adds new constraint.

- Simplest case: turbulence with adiabatic electrons
  - density profile cannot change
  - with no other constraints, the distribution function of each species is of the form
    \[
    f_{a0}(x) = F_{a0}[\epsilon(r, v) + \kappa(r)]
    \]