Conservative discontinuous Galerkin schemes for Boltzmann-Maxwell equations

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Wish to solve the Vlasov-Maxwell system:

\[
\begin{align*}
\frac{\partial f_s}{\partial t} + \nabla \cdot \alpha f_s &= 0 & \alpha &= (v, \alpha) (E + v \times B)/m_s \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0 & \nabla \cdot E &= \frac{1}{\epsilon_0} \frac{\partial \phi}{\partial t} \\
\epsilon_0 \mu_0 \frac{\partial E}{\partial t} - \nabla \times B &= -\mu_0 \mathbf{J} & \nabla \cdot B &= 0
\end{align*}
\]

Discretize with discontinuous Galerkin (DG) finite elements:

\[
\begin{align*}
M_0^s &= \int_{-\infty}^{\infty} dv \ f_s(t, x, v) \\
M_{1,i}^s &= \int_{-\infty}^{\infty} dv \ v_i f_s(t, x, v) \\
M_2^s &= \int_{-\infty}^{\infty} dv \ v^2 f_s(t, x, v)
\end{align*}
\]
Conservative DG Vlasov scheme\textsuperscript{1}

Discretize phase-space with a mesh $\mathcal{T}$ of cells $K_j \in \mathcal{T}$, $j = 1, 2, \cdots, N$.

Introduce the piecewise polynomial space $\mathcal{V}_h^p = \{v : v|_{K_j} \in \mathbf{P}_p^p, \forall K_j \in \mathcal{T}\}$ of degree $p$.

Typically choose Serendipity $\mathbf{P}_p^p$, spanned by the basis functions $\psi_k$, $k \in \{0, 1, \ldots, p\}$.

e.g. for 1x2v, $p = 1$

$$\psi_k \in \sqrt{\frac{3}{8}} \left\{ \frac{1}{\sqrt{3}}x, v_x, v_y, \sqrt{3}xv_x, \sqrt{3}xv_y, \sqrt{3}v_xv_y, 3xv_xv_y \right\}$$

We wish to find $f_h = f_k \psi_k$ such that:

$$\int_{K_j} d\mathbf{z} \psi_k \frac{\partial f_h}{\partial t} + \oint_{\partial K_j} dS \psi_k^- n \cdot \hat{\mathbf{F}} - \int_{K_j} d\mathbf{z} \nabla \cdot \alpha_h f_h = 0$$

Similar weak form treatments of Maxwell’s induction equations.

We wish to find $f_h = f_k \psi_k$ such that:

$$\int_{K_j} dz \psi_k \frac{\partial f_h}{\partial t} + \oint_{\partial K_j} dS \psi_k^- n \cdot \hat{F} - \int_{K_j} dz \nabla_z \cdot \alpha_h f_h = 0$$

The discrete scheme

- Conserves total number of particles.
- Has $\sum_j \frac{d}{dt} \int_{K_j} dz (-f_h \ln f_h) \geq 0$ if $f_h > 0$.
- Conserves total energy if central fluxes are used for Maxwell’s equations, and $v^2 \in V_h^p$.

1x1v test with strong initial asymmetric momentum

$m_p/m_e = 1836$
$v_{th,e} = 0.1c$
$T_p/T_e = 1$

Also wish to solve the Boltzmann equation

\[
\frac{\partial f_s}{\partial t} + \nabla_z \cdot \alpha f_s = C[f_s]
\]

In this talk…

- Conservative DG scheme for the Lenard-Bernstein operator (LBO).
- Weak equalities.
- Relaxation tests.
- Full Boltzmann-Maxwell tests.
Boltzmann equation with Lenard-Bernstein Operator (LBO)

\[ \frac{df}{dt} = \frac{\partial}{\partial v_i} \nu (v_i - u_i) f + \frac{\partial}{\partial v_i \partial v_i} \nu v_{ih}^2 f \]

In this talk
\[ \nu \neq \nu(v) \]

In the continuous, infinite velocity space this equation satisfies:

- Conservation of particles (\( M_0 \))
- Momentum (\( M_{1,i} \))
- Energy (\( M_2 \))

provided
\[ M_0 u_i = M_{1,i} \]
\[ d_v M_0 v_{ih}^2 + M_{1,i} u_i = M_2 \]

\( d_v \): number of velocity dimensions

- Monotonic increase of entropy:
\[ \sum_j \frac{d}{dt} \int_{K_j} \int_{m} dz (-f_h \ln f_h) \geq 0 \]

Note the Fokker-Planck operator in Rosenbluth form:

\[ \frac{df_s}{dt} = -\frac{\partial}{\partial v_i} \langle \Delta v_i \rangle_s f_s + \frac{1}{2} \frac{\partial}{\partial v_i \partial v_j} \langle \Delta v_i \Delta v_j \rangle_s f_s. \]
Can we construct a discrete DG scheme with the same conservative properties?
Consider the 1v collision term only for now:

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \nu (v-u) f + \frac{\partial^2}{\partial v^2} \nu v^2_{th} f \quad \nu \neq \nu(v)
\]

**Weak form**

Multiply by basis function \( \psi_k(z) \) and integrate over phase space in cell \( K_j \):

\[
\int_{K_j} dz \psi_k \frac{\partial f}{\partial t} = \int_{\partial K_j} dS \left\{ \psi_k \left[ (v-u) f + v^2_{th} \frac{\partial f}{\partial v} \right] - \frac{\partial \psi_k}{\partial v} v^2_{th} f \right\} ^{v_{j+1/2}}_{v_{j-1/2}}
- \int_{K_j} dz \left[ \frac{\partial \psi_k}{\partial v} (v-u) f - \frac{\partial^2 \psi_k}{\partial v^2} v^2_{th} f \right]
\]

Sum over all cells and...

- Set \( \psi_k = 1 \)
- Set \( \psi_k = v \)
- Set \( \psi_k = v^2 \)

\[
\sum_j \int_{K_j} dz \frac{\partial f}{\partial t} = \nu \sum_j \int_{\partial K_j} dS \left[ (v-u) f + v^2_{th} \frac{\partial f}{\partial v} \right] ^{v_{j+1/2}}_{v_{j-1/2}} = 0
\]

\[
M_0 u - v^2_{th} f \Big|_{v_{\min}}^{v_{\max}} = M_1
\]

\[
M_1 u + v^2_{th} \left( M_0 - v f \Big|_{v_{\min}}^{v_{\max}} \right) = M_2
\]
In arbitrary dimensions $M_0, M_{1,i}$ and $M_2$ are conserved as long as $u_i$ and $v_{th}^2$ are computed via

$$M_0 u_i - v_{th}^2 \int_{\partial \Omega_{v_i}} dS_i f \bigg|_{v_i,max}^{v_i,min} = M_{1,i}$$

$$M_{1,i} u_i + v_{th}^2 \left[ M_0 - \int_{\partial \Omega_{v_i}} dS_i (v_i f) \bigg|_{v_i,max}^{v_i,min} \right] = M_2$$

But we need to compute the expansion coefficients in

$$u_i = u_{i,k} \psi_k$$

$$v_{th}^2 = v_{th,k}^2 \psi_k$$
Motivation

Given the inner product on interval $I$

$$\langle f, g \rangle_I = \int_I dz \ f \ g$$

weak equality is defined as

$$f \overset{\cdot}{=} g \quad \iff \quad \langle f - g, \psi_k \rangle_I = 0$$

Weak equalities define, for example, our moments:

$$M_0 \overset{\cdot}{=} \int_{-\infty}^{\infty} dv \ f(t, x, v)$$

Use weak moments in conservation proofs to arrive at

$$M_0 u - v_{\text{max}}^2 \int_{v_{\text{min}}}^{v_{\text{max}}} = M_1$$

Obtain the (weak division) problem:

$$u M_0 \overset{\cdot}{=} M_1 \quad \implies \quad u_t M_{0,m} \int_I dx \ \psi_l \psi_m \psi_k = M_{1,l} \int_I dx \ \psi_l \psi_k$$

$$(E_M \cdot M_0) \cdot u = M \cdot M_1$$

$$u = (E_M \cdot M_0)^{-1} \cdot M \cdot M_1$$

Solve this linear system to compute velocities.

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To conserve $M_{1,i}$ and $M_2$ we must actually solve the (weak) system:

$$M_0 u_i - v_{th}^2 \int_{\partial \Omega_{v_i}} dS_i f \Bigg|_{v_{i,min}}^{v_{i,max}} \equiv M_{1,i}$$

$$M_{1,i} u_i + v_{th}^2 \left[ M_0 - \int_{\partial \Omega_{v_i}} dS_i (v_i f) \Bigg|_{v_{i,min}}^{v_{i,max}} \right] \equiv M_2$$

For $p = 1$ in 3D this is a $20 \times 20$ system of equations.
Another important use of weak equalities: Recovery polynomial

Weak LBO scheme:

\[ \int_{K_j} dz \psi_k \frac{\partial f}{\partial t} = \int_{\partial K_j} dS \left\{ \psi_k \left[ (v - u) f + v_{th}^2 \frac{\partial f}{\partial v} \right] - \frac{\partial\psi_k}{\partial v} v_{th} f \right\} \bigg|_{v_{j+1/2}}^{v_{j-1/2}} \]

\[ - \int_{K_j} dz \left[ \frac{\partial\psi_k}{\partial v} (v - u) f - \frac{\partial^2\psi_k}{\partial v^2} v_{th}^2 f \right] \]

Need to evaluate derivatives of \( f \) at the cell boundary, but \( f \) is in general discontinuous there.

Consider two adjacent cells

Given the solution in two adjacent cells, \( f_L \) and \( f_R \), each an expansion of order \( p \), we can construct the recovery polynomial

\[ \hat{f} = \sum_{m=0}^{2p-1} \hat{f}_m x^m \]

with

\[ \hat{f} = f_L \]

\[ \hat{f} = f_R \]

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We can check that relaxation of an initial state behaves as expected.

\[
f(t=0, x, v_x) = \begin{cases} 
    n(x)/(2v_0(x)) & |v_x| < v_0(x) \\
    0 & |v_x| \geq v_0(x).
\end{cases}
\]

\[n(x) = 1 + 0.4 \cos(2\pi x) \quad v_0(x) = n(x)/\sqrt{3}\]

Use \( p = 2 \) and \( \nu = 0.01 \) to solve

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \nu(v - u) f + \frac{\partial^2}{\partial v^2} \nu v_{th}^2 f
\]
Can also check $M_1$ conservation, and in higher dimensions.

Consider a 2x2v distribution with a large bump in its tail.
Conservation in Boltzmann-Maxwell system is also preserved.

Consider a 2x2v distribution with a large bump in its tail:

1x1v test has initial Maxwellian with strong asymmetric momentum:

\[ n(x, t = 0) = 1 + \exp \left[ -\beta (x - x_m)^2 \right] \]
\[ x_m = L_x/4 \]
\[ \beta = \begin{cases} 
0.75/d_e^2 & x < x_m \\
0.075/d_e^2 & x > x_m 
\end{cases} \]
\[ u(x, t = 0) = v_{th} \]
\[ v_{th,e} = 0.1c \]
\[ T_p/T_e = 1 \]
\[ m_p/m_e = 1836 \]
Landau damping of a Langmuir wave

Set up a 1x1v plasma with a small perturbation in the electron density and electric field:

\[ n_e(x) = 1 + 10^{-4} \cos\left(\frac{x}{2\lambda_D}\right) \quad n_p(x) = 1 \]

\[ E_x(x) = -\frac{1}{\varepsilon_0} |e| 2 \times 10^{-4} \lambda_D \sin\left(\frac{x}{2\lambda_D}\right) \]

\[ v_{th,e} = 0.1c \]
\[ T_p/T_e = 1 \]
\[ m_p/m_e = 1836 \]

Magnetic pumping heating in the solar wind$^1$

**VPIC 2D simulation**
- electron-ion collisions only.
- Monte Carlo Coulomb collisions.
- $\omega_{pe} = \Omega_e$
- $m_i/m_e = 100$
- Drive current sheets.

**Gkeyll 1x3v simulation**
- e-e and ion-ion collisions only.
- Lenard-Bernstein operator.
- $\omega_{pe} = 0.1\Omega_e$
- Drive oscillating B-field.


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Summary

• A particle, momentum and energy conserving DG scheme is possible for the LBO.

• Conservation requires the solution to a weak system of equations to obtain $\mathbf{u}$ and $v_{th}^2$.

• Our scheme uses a recovery polynomial to evaluate derivatives at cell boundaries.

• LBO equation evolves $f$ to the respective Maxwellian while conserving $M_0$, $M_{1,i}$ and $M_2$.

• Vlasov terms retain particle and energy conservation.

• Begun exploring Lichko's magnetic pumping heating mechanism. In our 1x3v model heating is much weaker and does not follow the same trend with $\nu$. 

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